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Reg. No. : ....

Name : ....

Fifth Semester B.Sc. Degree Examination, February 2021

First Degree Programme under CBCSS

Mathematics

Core Course VII

MM 1543 — ABSTRACT ALGEBRA – GROUP THEORY

(2018 Admission - Regular)

Time : 3 Hours Max. Marks : 80

SECTION – I

All the first **10** questions are compulsory. They carry **1** mark each.

1. Describe the set of symmetries of an equilateral triangle.

2. What is the inverse of 13 in $U(14)$?

3. Determine all elements of finite order in R^*, the group of non zero real numbers under multiplication.

4. What is the order of the following permutation (124)(35)?

5. Prove that S_n is non-Abelian for all $n \geq 3$.

6. Why the set of odd permutations in S_n is not a subgroup?

P.T.O.

7. Show that a group of prime order is cyclic.

8. If a group G has a unique subgroup H of some finite order, show that H is normal in G.

9. Define Group Homomorphism.

10. Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and $\ker\phi = \{0, 10, 20\}$. If $\varphi(23) = 9$, determine all elements that map to 9.

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions from among the questions **11** to 26. These questions carry **2** marks each.

11. Let a, b and c be elements of a group. Solve the equation $axb = c$ for x. Solve $a^{-1}xa = c$ for x.

12. Suppose that H is a non empty subset of a group G that is closed under the group operation and has the property that if a is not in H then a^{-1} is not in H. Is H a subgroup?

13. Prove that a group with two elements of order 2 that commute must have a subgroup of order 4.

14. Suppose that H is a proper subgroup of Z under addition and that H contains 12, 30, and 54. What are the possibilities for H?

15. For $n > 1$ prove that A_n has order $n!/2$.

16. Find $Aut(Z)$.

17. Show that Z has infinitely many subgroups isomorphic to Z.

18. Prove or disprove that $U(20)$ and $U(24)$ are isomorphic.

19. If G is a finite group and H is a subgroup of G, then show that $|H|$ divides $|G|$.

20. State and prove Fermats Little Theorem.

21. Show that a subgroup H of G is normal in G if and only if $xHx^{-1} \in H$ for all x in G.

22. Let ϕ be an isomorphism from a group G onto a group G. Prove that if H is a normal subgroup of G, then $\phi(H)$ is a normal subgroup of $\overline{G}$.

23. If ϕ is a homomorphism from Z_{30} onto a group of order 5, determine the kernel of ϕ.

24. Suppose that ϕ is a homomorphism from $U(40)$ to $U(40)$ and that $\ker \phi = \{1, 9, 17, 33\}$. If $\phi(11) = 11$, find all elements of $U(40)$ that map to 11.

25. Show that every normal subgroup of a group G is the kernel of a homo-morphism of G.

26. Determine all homomorphisms from Z_n to itself.

(8 × 2 = 16 Marks)

SECTION – III

Answer **any six** questions from among the questions 27 to 38. These questions carry **4** marks each.

27. Prove that the dihedral group of order 6 does not have a subgroup of order 4.

28. If d is a positive divisor of n, show that the number of elements of order d in a cyclic group of order n is $\phi(d)$.

29. Suppose that a cyclic group G has exactly three subgroups: G itself, $\{e\}$, and a subgroup of order 7. What is $|G|$? What can you say if 7 is replaced with p where p is a prime?

30. Prove that Q^{*}, the group of positive rational numbers under multiplication, is isomorphic to a proper subgroup.

31. Suppose that $\phi : Z_{20} \to Z_{20}$ is an automorphism and $\phi(5) = 5$. What are the possibilities for $\phi(x)$?

32. Show that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

33. For two finite subgroups H and K of a group, show that $|HK| = |H||K|/|H \cap K|$.

34. State and prove the Orbit-Stabilizer Theorem.

35. Show that Q, the group of rational numbers under addition, has no proper subgroup of finite index.

36. Suppose that there is a homomorphism from a finite group G onto Z_{10}. Prove that G has normal subgroups of indexes 2 and 5.

37. State and prove the First Isomorphism Theorem.

38. Suppose that ϕ is a homomorphism from Z_{36} to a group of order 24.

(a) Determine the possible homomorphic images.

(b) For each image in part a, determine the corresponding kernel of ϕ.

(6 × 4 = 24 Marks)

SECTION – IV

Answer **any two** questions from among the questions 39 to 44. These questions carry **15** marks each.

39. Prove that the center of a group G is a subgroup of G and determine the centers of the dihedral groups.

40. (a) Prove that an infinite group must have an infinite number of subgroups.

    (b) Suppose that a and b are group elements that commute and have orders m and n. If $\langle a \rangle \cap \langle b \rangle = \{e\}$, prove that the group contains an element whose order is the least common multiple of m and n. Show that this need not be true if a and b do not commute.

41. (a) Show that every group is isomorphic to a group of permutations.

    (b) Determine $Inn(D_4)$.

42. Compute $Aut(Z_{10})$ and show that it is isomorphic to $U(10)$.

43. Let G be a group of order 2p, where p is a prime greater than 2. Then show that G is isomorphic to Z_{2p} or D_p.

44. Let f be a homomorphism from a group G to a group $\overline{G}$ and let H be a subgroup of G. Prove the following :

    (a) $\phi(H) = \{\phi(h) \mid h \in H\}$ is a subgroup of G

    (b) If H is cyclic, then $\phi(H)$ is cyclic.

    (c) If H is Abelian, then $\phi(H)$ is Abelian.

    (d) If H is normal in G, then $\phi(H)$ is normal in $\phi(G)$.

    (e) If $|\ker \phi| = n$, then ϕ is an n-to-1 mapping from G onto $\phi(G)$.

(f) If $|H| = n$, then $|\phi(H)|$ divides n.

(g) If K is a subgroup of $\overline{G}$, then $f^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\}$ is a subgroup of G.

(h) If $\overline{K}$ is a normal subgroup of $\overline{G}$, then $f^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\}$ is a normal subgroup of G.

(i) If ϕ is onto and $\ker \phi = \{e\}$, then ϕ is an isomorphism from G to $\overline{G}$.

(2 × 15 = 30 Marks)