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Reg. No. : ....

Name : ....

Fifth Semester B.Sc. Degree Examination, February 2021

First Degree Programme Under CBCSS

Mathematics

Core Course VI

MM 1542 COMPLEX ANALYSIS I

(2018 Admission-Regular)

Time : 3 Hours Max. Marks : 80

SECTION – I

All the first **ten** questions are compulsory. They carry **1** mark each.

1 Write $1+\sqrt{3}i$ in polar form.

2 Which of the points $i, 2-i$, and -3 is farthest from the origin?

3 Compute $(1+i)^{24}$.

4 Define analyticity of a complex-valued function on an open set.

5 Let $f(z)=e^z-(1+z+z^2/2+z^3/6)$. Find $f^{(3)}(0)$.

6 Express $\cos(1-i)$ in the form $a+bi$.

7 Evaluate $Log(\sqrt{3}+i)$

P.T.O.

8. Define a contour.

9. Compute $\int_{\Gamma} \bar{z}\, dz$ where Γ is the circle $|z| = 2$ traversed once counterclock-wise.

10. If $P(z)$ is a polynomial and Γ is any closed contour, explain why $\int_{\Gamma} P(z)\, dz = 0$.

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions from among the questions **11** to **26**. These questions carry **2** marks each.

11. Find the most general harmonic polynomial of the form $ax^2 + bxy + cy^2$.

12. Prove that the function $f(z) = e^z$ is entire and find its derivative.

13. Prove that the function $g(z) = z$ is continuous on the whole plane.

14. Show that the function Arg z is discontinuous at each point on the non-positive real axis.

15. Find all values of $(1-i)^{3/2}$.

16. Find all the poles and their multiplicities for $R(z) = \dfrac{(3z + 3i)(z^2 - 4)}{(z-2)(z^2+1)^2}$.

17. A polynomial $p(z)$ of degree 4 has zeros at the points 1, $3i$ and $-3i$ of respective multiplicities 2, 1 and 1. If $p(1) = 80$, find $p(z)$.

18. Show that if the rational function $R(z)$ has a pole of order m at z_0, then its derivative $R'(z)$ has a pole of order $m + 1$ at z_0.

19. Show that the function $f(z) = (x^2 + y) + i(y^2 - x)$ is not analytic at any point.

20. Evaluate $\int_{\Gamma} (3z^2 - 5z + i)\, dz$ along the line segment from $z = i$ to $z = 1$.

21. Prove that $\sin z = 0$ if and only if $z = k\pi$ where k is an integer.

22. Prove that the function e^z is one-to-one on any open disk of radius π.

23. True or false: If f is analytic at each point of a closed contour, then $\int_\Gamma f(z)dz = 0$.

24. Show that if C is a positively oriented circle and z_0 lies outside C, then $\int_C \frac{dz}{z - z_0} = 0$.

25. Compute $\int_\Gamma \frac{1}{z - z_0} dz$ where Γ is the circle $|z - z_0| = r$ traversed twice in the counterclockwise direction starting from the point $z_0 + r$.

26. When do you say that the loop Γ_0 is continuously deformable to the loop Γ_1 in the domain D?

(8 × 2 = 16 Marks)

SECTION – III

Answer any **six** questions, from among the questions **27** to **38**. These questions carry **4** marks each.

27. Compute the integral $\int_0^{2\pi} \cos^4 \theta \, d\theta$.

28. Let $f(z)$ be defined by $f(z) = \begin{cases} \dfrac{2z}{z+1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$.

At which points does $f(z)$ have a finite limit, and at which points is it continuous? Which of the discontinuities of $f(z)$ are removable?

29. Show that $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but is nowhere analytic.

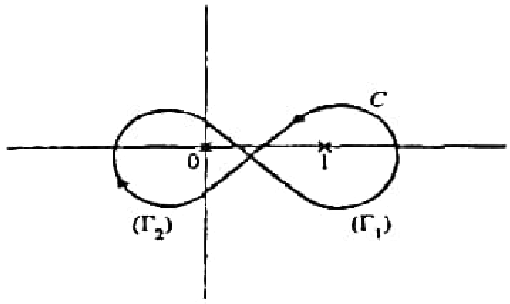
30. Determine a branch of $f(z) = \log(z^3 - 2)$ that is analytic at $z = 0$, and find $f(0)$ and $f'(0)$.

31. Prove that there exists no function $F(z)$ analytic in the annulus $D: 1 < |z| < 2$ such that $F'(z) = 1/z$ for all z in D.

32. Prove that any branch of log z is analytic in its domain and has derivative 1/z.

33. Find the maximum value of $|z^2 + 3z - 1|$ in the disk $|z| \leq 1$

34. Compute $\int_C \frac{2z+1}{z(z-1)^2} dz$ along the figure-eight contour C sketched below:



35. Find an upper bound for $\left| \int_\Gamma \frac{e^z}{z^2+1} dz \right|$ where Γ is the circle $|z| = 2$ traversed once in the counterclockwise direction.

36. Evaluate $\int_\Gamma \frac{e^z}{(z^2+1)^2} dz$ where Γ is the circle $|z| = 3$ traversed once counterclockwise.

37. If f is analytic in the annulus $1 \leq |z| \leq 2$ and $|f(z)| \leq 3$ on $|z| = 1$ and $|f(z)| \leq 12$ on $|z| = 2$, prove that $|f(z)| \leq 3|z|^2$ for $1 \leq |z| \leq 2$.

38. Write the polynomial, $p(z) = z^3 + 3z + 4$ in the Taylor form, centered at z = 2.

(6 × 4 = 24 Marks)

SECTION – IV

Answer **any two** questions from among the questions **39** to **44**. These questions carry **15** marks each.

39\. (a) Let $f(z) = \begin{cases} \dfrac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2 + y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Show that the Cauchy-Riemann equations hold at $z = 0$ but that f is not differentiable at this point.

(b) Suppose that $f(z)$ is analytic and nonzero in a domain D. Prove that $\ln|f(z)|$ is harmonic in D.

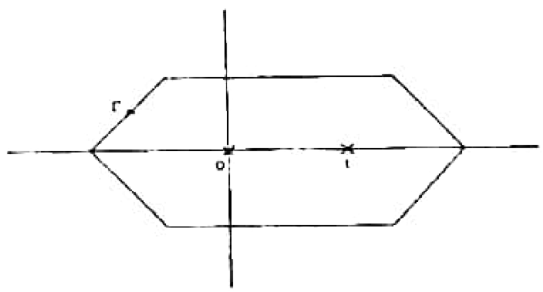
40\. (a) Show that the equation $e^z = 1$ holds if and only if $z = 2k\pi i$, where k is an integer. Also show that the equation $e^{z_1} = e^{z_2}$ holds if and only if $z_1 = z_2 + 2k\pi i$, where k is an integer.

(b) Define a branch of $(z^2 - 1)^{1/2}$ that is analytic in the exterior of the unit circle, $|z| > 1$.

41\. (a) State and prove Cauchy's Integral Formula.

(b) Show that every nonconstant polynomial with complex coefficients has at least one zero.

42\. (a) Compute $\int_\Gamma \frac{3z - 2}{z^2 - z} dz$ where Γ is the simple closed contour indicated in the following figure



(b) Suppose that f is analytic inside and on the unit circle $|z| = 1$. Prove that if $|f(z)| \leq M$ for $|z| = 1$, then $|f(0)| < M$ and $|f'(0)| \leq M$. What estimate can you give for $|f^{(n)}(0)|$?

43. (a) Compute the integral $\int_{C_r} (z - z_0)^n dz$, with n an integer and C_r the circle $|z - z_0| = r$ traversed once in the counterclockwise direction.

(b) Show that the function Log z is analytic in the domain D^* consisting of all points of the complex plane except those lying on the nonpositive real axis. Also show that $\frac{d}{dz} Log\, z = \frac{1}{z}$ for all $z \in D^*$.

44. Let g be continuous on the contour Γ and for each z not on Γ, set $G(z) = \int_{\Gamma} \frac{g(\zeta)}{\zeta - z} dz$. Show that the function G is analytic at each point not on Γ, and its derivative is given by $G'(z) = \int_{\Gamma} \frac{g(\zeta)}{(\zeta - z)^2} dz$.

(2 × 15 = 30 Marks)
