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Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Degree Examination, March 2021

First Degree Programme under CBCSS

Mathematics

Core Course – X

MM 1642 – COMPLEX ANALYSIS II

(2018 Admission Regular)

Time : 3 Hours Max. Marks : 80

SECTION – A

All the first **10** questions are compulsory. They carry **1** mark each.

1. Write the power series representation of $f(z) = \frac{1}{z-1}$ in powers of z.

2. If f(z) is analytic inside a circle C with centre at z_0 and $f(z) = \sum_{i=0}^{r} a_k (z - z_0)^k$. What is the value of a_k?

3. What is the order of the zero of $z(e^z - 1)$?

4. Classify the singularity at $z = 0$ of the function $f(z) = \frac{\sin z}{z^5}$.

5. Define Cauchy principal value of the improper integral $\int_{-r}^{r} x dx$.

P.T.O.

6. State Jordan's lemma.

7. State Reimann mapping theorem.

8. Define Mobius transformation.

9. What type of singularity the function $e^{1/z}$ has at $z = 0$?

10. Find the residue at $z = 0$ for the function $f(z) = \frac{1}{z + z^2}$.

(10 × 1 = 10 Marks)

SECTION – B

Answer **any eight** questions (11-26) Each question carries **2** marks

11. Using comparison test, show that the series $\sum_{k=1}^{\infty}\left(\frac{1}{k^2 + 1}\right)$ converges.

12. Find the circle of convergence of the power series $\sum_{k=0}^{\infty}\frac{(z-2)^k}{3^k}$.

13. If the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k Z^k$ is R, find the radius of convergence of $\sum_{k=0}^{\infty} k^3 a_k z^k$.

14. Find the Maclaurin series of $\sinh z$.

15. Find the residue of the function $f(Z) = \frac{z^3 + z^2}{(z-1)^3}$, at $z = 1$.

16. Determine the zeros and their order of the function $f(z) = z \sin z^2$.

17. Determine the order of the pole and residue of the function $\frac{\sinh z}{z^4}$ at $z = 0$.

18 Write down the principal part of the function $z\exp\left(\frac{3}{z}\right)=\left(z+3+\frac{3^2}{2!\,z}+\ldots\right)$ at its isolated singular point and determine the nature of the singularity.

19 Find the residue of $f(z)=\tan z$ at each of its singular points.

20. Show that $\int_C \exp\left(\frac{1}{z^2}\right)dz=0$, where C is positively oriented circle $|z|=1$.

21. Evaluate $\int_C \cot z\,dz$, where C denote the positively oriented circle $|z|=2$.

22. If z_0 is a pole of $f(z)$, show that $\lim_{z\to z_0} f(z)=\infty$.

23. Find $\sum_1^\infty \frac{1}{n^2}$.

24. Show that a Mobius transformation, not the identity, has at most two fixed points.

25. Define the cross ratio of the four points z, z_1, z_2, z_3 and find the cross ratio (z, -1, 0, 1).

26. Find the map of the circle $|z|=3$ under the transformation $w=2z$.

(8 × 2 = 16 Marks)

SECTION – C

Answer **any six** questions (27-38). Each question carries **4** marks.

27. Find a power series representing the function $f(z)=\frac{1}{z^2}$ about $z=2$. Also find the radius of convergence.

28. State and prove Weierstrass M-test.

29. Find the Laurent's series expansion for the function $f(z) = \frac{1}{z(z-1)}$ valid for $1 < |z-2| < 2$.

30. Let f be analytic at z_0. Prove that f has a zero of order m for $f(z)$ at z_0 if and only if f can be written as $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

31. Find the singularities of the function $f(z) = \frac{4-3z}{z(z-1)(z-2)}$ and hence find the corresponding residues.

32. Evaluate $\int_C \frac{z^3 dz}{(z-2)^2}$ where C is the positively oriented circle $|z-1| = 2$.

33. If $f(z)$ has a pole of order m at $z = a$, then show that $\operatorname{Res}_{z=a} f(z) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}$ at $z = a$.

34. Use residues to prove that $\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$.

35. Find the image of the unit circle $|z| = 1$ under the linear fractional transformation $f(z) = \frac{z+2}{z-1}$.

36. Find the Mobius transformation that maps 0, i, ∞ onto 0, 1, 2.

37. Prove that the transformation $w = f(z) = \frac{1}{z}$ maps circles passing through the origin onto lines not passing through the origin.

38. Prove that a cross ratio is invariant under linear fractional transformation.

(6 × 4 = 24 Marks)

SECTION – D

Answer **any two** questions (39-44). Each question carries **15** marks

39. (a) Find the zeros of the analytic function $f(z) = z \sin z^2$.

    (b) Describe the three types of isolated singular points.

    (c) Determine the order of the rational function $f(z) = \dfrac{2z+5}{(z-1)(z+5)(z-2)^4}$.

40. (a) Use Cauchy's residue theorem to evaluate $\int_C \dfrac{z+1}{z^2-2z}\,dz$ around the circle $|z| = 3$ in the positive sense.

    (b) Use residue theorem to prove that $\int_0^{\infty} \dfrac{1}{1+x^6}\,dx = \dfrac{\pi}{3}$.

41. (a) State and prove Cauchy's residue theorem.

    (b) Compute P.V $\int_{-\infty}^{\infty} \dfrac{x \cos x}{x^2+9}\,dx$.

42. (a) Find the singularities of the function $f(z) = \dfrac{z^2 - z}{(z+1)^2(z^2+4)}$ and hence find the corresponding residues.

    (b) Show that $\int_{-\infty}^{\infty} \dfrac{\cos x}{(x^2+1)^2}\,dx = \dfrac{\pi}{e}$.

43. (a) Show that $\int_0^{2\pi} \dfrac{d\theta}{1+a\cos\theta} = \dfrac{2\pi}{\sqrt{1-a^2}} \ (-1 < a < 1)$.

    (b) Use residues to find the Cauchy principle value of $\int_{-\infty}^{\infty} \dfrac{dx}{(x^2-3x+2)(x^2+1)}$.

44. (a) Find all the points where the mapping $f(z) = \sin z$ is conformal.

(b) Find the Mobius transformation that maps $-1, i, 1$ onto $0, i, \infty$.

(c) Discus the image of the circle $|z-2|=1$ under the transformation $w = \dfrac{z-4}{z-3}$.

(2 × 15 = 30 Marks)
