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Reg. No. : .....

Name : .....

**Sixth Semester B.Sc. Degree Examination, March 2021**

**First Degree Programme under CBCSS**

**Mathematics**

**Core Course IX**

**MM 1641 : REAL ANALYSIS - II**

**(2015-2017 Admission)**

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the first **ten** questions are compulsory. Each question carries **1** mark.

- 1 State  $\epsilon - \delta$  definition of continuity of a function.
2. State Discontinuity Criterion.
3. Determine the points of continuity of the function defined by  $f(x) = [[\sin x]]$ ,  $x \in \mathbb{R}$  where  $[[x]]$  denotes the greatest integer less than or equal to  $x$ .
4. Give an example of a function  $f : [0, 1] \rightarrow \mathbb{R}$  that is discontinuous at every point of  $[0, 1]$  but  $|f|$  is continuous on  $[0, 1]$ .

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5. State Boundedness theorem.
6. Give an example of a function which is monotone but not continuous.
7. Find the points of relative extrema of the function  $f(x) = |x^2 - 1|$  for  $-4 \leq x \leq 4$ .
8. State Cauchy Mean value theorem.
9. Evaluate  $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$ .
10. Define Riemann integrable function.

**(10 × 1 = 10 Marks)**

### SECTION – II

Answer **any eight** questions from this section. Each question carries **2** marks.

11. Discuss the continuity of the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  at  $x = 0$ , defined by

$$F(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \end{cases}$$

12. Show that rational function is continuous at every real number for which it is defined.
13. Give an example of functions  $f$  and  $g$  that are both discontinuous at a point  $c$  in  $\mathbb{R}$  such that the sum  $f + g$  is continuous at  $c$ .

14. Show that if  $f : I \rightarrow \mathbb{R}$  has a derivative at  $c \in I$  then  $f$  is continuous at  $c$ .
15. Show that the function  $f(x) = x^{1/3}$ ,  $x \in \mathbb{R}$  is not differentiable at  $x = 0$ .
16. If  $a > 0, b > 0$  and  $\alpha$  be any real number with  $0 < \alpha < 1$  then prove that  $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b$  where equality holds if and only if  $a = b$ .
17. Let  $f$  and  $g$  be defined on  $[a, b]$ , let  $f(a) = g(a) = 0$ , and let  $g(x) \neq 0$  for  $a < x < b$ . If  $f$  and  $g$  are differentiable at  $a$  and if  $g'(a) \neq 0$ , then prove that limit of  $\frac{f}{g}$  at  $a$  exists and is equal to  $\frac{f'(a)}{g'(a)}$ .
18. Evaluate  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ .
19. Let  $f(x) = x$  for  $x$  in  $[0, 1]$ . Prove that  $f \in \mathcal{R} [0, 1]$  and  $\int_0^1 f(x) dx = \frac{1}{2}$ .
20. Using substitution theorem, evaluate  $\int_1^4 \frac{\cos \sqrt{t}}{\sqrt{t}} dt$ .
21. Prove that the Dirichlet function  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$  is not Riemann integrable.
22. Find  $\int_{-10}^{10} \text{sgn}(x) dx$ .

**(8 × 2 = 16 Marks)**

### SECTION – III

Answer **any six** questions from this section. Each question carries **4** marks.

23. Let  $A = \{x \in \mathbb{R}; x > 0\}$ . Define  $h: A \rightarrow \mathbb{R}$  by

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ is rational} \end{cases}$$

Where  $m$  and  $n$  are natural numbers having no common factors except 1. Show that  $h$  is continuous at every irrational number in  $A$ , and is discontinuous at every rational number in  $A$ .

24. Let  $A, B \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$  be continuous on  $A$ , and let  $g: B \rightarrow \mathbb{R}$  be continuous on  $B$ . If  $f(A) \subseteq B$ , prove that the composite function  $g \circ f: A \rightarrow \mathbb{R}$  is continuous on  $A$ .
25. Prove that if  $f$  and  $g$  are two functions differentiable at  $c$  then their product  $fg$  is differentiable at  $c$ .
26. Given that the function  $f(x) = x^n$ ,  $x \geq 0$ ,  $n \in \mathbb{N}$ , has an inverse  $g$  on  $(0, \infty)$ , find  $g'(y)$  for  $y > 0$ .
27. State and prove Mean value theorem.
28. Let  $f: I \rightarrow \mathbb{R}$  be differentiable on the interval  $I$ . Prove that  $f$  is increasing on  $I$  if and only if  $f'(x) \geq 0$  for all  $x \in I$ .

29. State and prove Darboux's theorem.
30. If  $f \in \mathcal{R}[a, b]$ , then prove that the value of the integral is uniquely determined.
31. Prove that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$ , then  $f \in \mathcal{R}[a, b]$ .

**(6 × 4 = 24 Marks)**

SECTION – IV

Answer **any two** questions from this section. Each question carries **15** marks.

- 32 (a) State and prove Maximum-Minimum theorem. **8**
- (b) State and prove chain rule of differentiation. **7**
- 33 (a) Let  $I$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be strictly monotone on  $I$ . Let  $J = f(I)$  and let  $g : J \rightarrow \mathbb{R}$  be the function inverse to  $f$ . If  $f$  is differentiable on  $I$  and  $f'(x) \neq 0$  for  $x \in I$ , then prove that  $g$  is differentiable on  $J$  and  $g' = \frac{1}{f' \circ g}$ . **8**
- (b) State and prove Caratheodory's theorem. **7**
34. (a) State and prove L'Hospital's Rule-I. **10**
- (b) If  $f$  and  $g$  are in  $\mathcal{R}[a, b]$  and  $f(x) \leq g(x)$  for all  $x$  in  $[a, b]$  then show that
- $$\int_a^b f(x) dx \leq \int_a^b g(x) dx. \quad \text{5}$$

35. (a) If  $f \in \mathcal{R}[a, b]$ , prove that  $f$  is bounded on  $[a, b]$ . 7

(b) State and prove Fundamental theorem of Calculus (Second Form). 8

**(2 × 15 = 30 Marks)**

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